Common Pairings and Invariant Structures for Companion Matrix Families with a Fixed Factor

Aaron Nagy[a,*]

[a]*Independent researcher, South Bend, Indiana, USA*

Abstract

Let s and t be monic polynomials over an arbitrary field, and let the residual factors in the Frobenius companion matrices C_{sq} and C_{tw} vary freely while s and t remain fixed. We classify structures that are common to the resulting families. The rectangular solutions of $C_{sq}^T H = H C_{tw}$ for every residual pair are naturally parametrized by $(K[x]/(\gcd(s,t)))^\vee$. A Frobenius functional on this quotient gives the rank and both ambient radicals of every common pairing; the possible ranks are exactly the degrees of divisors of $\gcd(s,t)$. The full family-wide condition is already detected by at most two residual specializations on each side. In the diagonal case every common solution is a symmetric Hankel form. For one family, the common invariant subspaces are the truncated ideals indexed by divisors of s. We give an inverse-image and triangular description of the generated algebra, recover its known dimension $n(n - \deg s) + \deg s$, determine its Jacobson radical and semisimple quotient, and show that the common commutant is scalar whenever the residual factor moves. Common covariant intertwiners exhibit a sharp asymmetry: motion in the target family forces vanishing, whereas a fixed target leaves a module isomorphic to $K[x]/(\gcd(s,t))$ with a complete divisor-controlled rank, kernel, and image stratification. Finite-field corollaries give the exact rank distributions.

Keywords: companion matrix, Hankel pairing, invariant subspace, generated algebra, cyclic module, polynomial greatest common divisor, finite-field rank distribution
2020 MSC: 15A24, 15A30, 15A63, 16S50

*Corresponding author
Email address: aaronnagy1987@gmail.com (Aaron Nagy)

1. Introduction

For a monic polynomial

$$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

let C_p be the Frobenius companion matrix in the monomial basis $1, x, \ldots, x^{n-1}$, so that C_p represents multiplication by x on $K[x]/(p)$. The linear algebra of a single companion matrix is the linear algebra of one cyclic $K[x]$-module. The object studied here is instead the residual family

$$\mathcal{F}_n(s) = \{C_{sq} : q \text{ monic}, \ \deg q = n - \deg s\},$$

where the polynomial factor s is held fixed and the residual factor q is allowed to vary freely. The phrase *fixed factor* is used literally and is unrelated to the number-theoretic notion of a fixed divisor of an integer-valued polynomial.

The passage from one cyclic operator to a residual family changes the basic questions. A bilinear form, invariant subspace, or endomorphism that exists for each member separately need not be independent of the residual factor. The structures that survive all residual deformations are controlled by the fixed quotient $K[x]/(s)$, but in different ways. Bilinear pairings between two families remember the greatest common divisor of their fixed factors. Common invariant subspaces remember the divisor lattice of the fixed factor. The generated algebra remembers the fixed quotient on top and becomes full on the forced ideal below it. Covariant intertwiners are more rigid and vanish as soon as the target residual factor moves.

The realization of a companion matrix as the shift on a polynomial quotient, and the accompanying duality between polynomial models, Hankel forms, and intertwining relations, belong to a classical framework developed extensively by Fuhrmann [1, 2]. Hankel relations for individual companion matrices and transformations between companion matrices are likewise classical; see, for example, [3, 4, 5]. The general single-matrix symmetrizer equation $A^T S = SA$ and its numerical solution also have an established literature [8]. Rank and Smith data for multiplication operators $f(C_g)$ are controlled by polynomial common divisors; Noferini and Williams give stronger Smith-form results in companion rings [9]. Thus neither the single-pair polynomial-model translation nor the quotient-ring rank calculation used in Theorem 3.4 is claimed as new. The point here is the intersection problem over two independently moving residual families, together with the exact pullback radicals in both ambient modules.

Algebras generated by finitely many companion matrices have also been studied. Guersenzvaig and Szechtman proved full-generation criteria in terms of resultants, the common-factor criterion for invariant subspaces, and a two-companion rank formula depending only on the degree of the polynomial gcd [10, 11]. Their rank

$$n + (n - m)(n - 1) = n(n - m) + m$$

agrees with the dimension obtained below when two residual specializations have gcd s. We therefore do not claim the generated-algebra dimension or the two-matrix generation phenomenon as new. The contribution pursued here is the simultaneous family classification: the common rectangular pairing space for two independently moving residual families, its explicit ambient radicals and finite detection certificate, the residual-family inverse-image and triangular realization together with its radical, and the field-independent sharp asymmetry between moving- and fixed-target intertwiners, including the complete fixed-target rank, kernel, and image strata. We also do not claim that the fixed-polynomial Hankel, multiplication-rank, or ideal correspondences are new.

The organization follows the surviving structure. Section 3 classifies common rectangular pairings and their finite-field rank distribution. Section 4 identifies the common invariant lattice. Section 5 determines the family algebra, its triangular form, radical, and commutant. Section 6 treats covariant intertwiners, where target motion produces a vanishing theorem and a fixed target produces a second divisor stratification.

The proofs use three elementary mechanisms. First, the companion shift columns force every common bilinear solution to be Hankel, while variation of the final column forces descent to the fixed quotient modules. Second, a Frobenius functional on a polynomial quotient converts pairing radicals into annihilator ideals. Third, differences of two residual specializations are rank-one maps; their left and right orbits under one companion matrix generate the full lower ideal of the family algebra and also prove the moving-target intertwiner no-go theorem. These mechanisms are kept explicit so that the endpoint cases of zero residual degree remain visible.

2. Companion modules and residual families

Throughout, K is a field and $n, N \geq 1$. For a finite-dimensional K-space A, write $A^\vee = \mathrm{Hom}_K(A, K)$ for its linear dual. Write

$$V_n = K[x]_{<n}.$$

For $f \in V_n$, let $\mathbf{c}_n(f) \in K^n$ denote its coefficient column in the ordered basis $1, x, \ldots, x^{n-1}$. For a monic polynomial p of degree n, the identification $V_n \cong K[x]/(p)$ transports multiplication by x to C_p. If d is monic with $\deg d \leq n$, define the truncated ideal

$$M_d^{(n)} = (d) \cap V_n = \mathrm{span}_K\{d, xd, \ldots, x^{n-\deg d-1}d\},$$

with $M_d^{(n)} = 0$ when $\deg d = n$.

Fix monic s, t with

$$\deg s = m \leq n, \qquad \deg t = \ell \leq N,$$

and write

$$p_q = sq, \qquad r_w = tw,$$

where q and w range over all monic polynomials of degrees $n - m$ and $N - \ell$, respectively. Degree-zero residual factors are interpreted as the constant polynomial 1.

3. Common rectangular pairings

A matrix $H \in K^{n\times N}$ represents a bilinear form $B_H : V_n \times V_N \to K$. We call it a common pairing of the two residual families if

$$C_{p_q}^T H = H C_{r_w} \qquad \text{for every residual pair } (q, w).$$

Its left and right radicals are

$$\ker_{\text{left}} H = \{f \in V_n : B_H(f, h) = 0 \ \forall h \in V_N\},$$

$$\ker_{\text{right}} H = \{h \in V_N : B_H(f, h) = 0 \ \forall f \in V_n\}.$$

Theorem 3.1 (gcd classification of common pairings)**.** *Let $g = \gcd(s,t)$ be monic. A matrix $H \in K^{n\times N}$ is a common pairing if and only if there is a unique functional $L \in (K[x]/(g))^\vee$ such that*

$$H_{ij} = L(x^{i+j} \bmod g), \qquad 0 \le i < n, \quad 0 \le j < N.$$

Consequently,

$$\dim_K\{H : C_{p_q}^T H = H C_{r_w} \ \forall q, w\} = \deg g,$$

and every common pairing has rank at most $\deg g$*.*

Proof. The shift columns of the companion matrices imply

$$B_H(xf, h) = B_H(f, xh)$$

whenever both shifted monomials remain in the two truncation windows. Hence there are moments $M_0, \dots, M_{n+N-2}$ such that

$$B_H(x^i, x^j) = M_{i+j};$$

in particular, H is rectangular Hankel.

Put $R = n - m$. The final left companion relation, obtained by applying the matrix equation to x^{n-1} in the left variable and x^j in the right variable, is

$$M(x^j s q) = 0, \qquad 0 \le j \le N-2,$$

where $M(x^a) = M_a$. If $R > 0$, write $q = x^R + \sum_{r=0}^{R-1} b_r x^r$. Varying each coefficient b_r and then using the monic relation gives

$$M(x^a s) = 0, \qquad 0 \le a \le R + N - 2.$$

These are exactly the moment identities required for B_H to annihilate $M_s^{(n)}$ in the left variable. If $R = 0$, then $M_s^{(n)} = 0$, so the same conclusion is vacuous. The analogous right companion relations, with the same fixed-endpoint convention, show that the pairing annihilates $M_t^{(N)}$ in the right variable. Thus B_H descends to a balanced pairing

$$K[x]/(s) \times K[x]/(t) \longrightarrow K.$$

Balanced pairings are dual to the tensor product, and the PID calculation

$$K[x]/(s) \otimes_{K[x]} K[x]/(t) \cong K[x]/(s,t) \cong K[x]/(g)$$

gives a unique $L \in (K[x]/(g))^\vee$.

Conversely, for such an L define

$$B_L(f,h) = L(fh \bmod g).$$

This pairing is balanced and kills both (s) and (t) because g divides both fixed factors. It therefore satisfies every companion relation in the two families. The dimension statement follows from $\dim K[x]/(g) = \deg g$. Finally, the pairing factors through the quotient maps $V_n \to K[x]/(g)$ and $V_N \to K[x]/(g)$, proving the rank bound. □

Corollary 3.2 (two-point detection of the full family)**.** *Choose monic residual factors q_0 and w_0. If $R = n - m > 0$, set $\mathcal{Q}_0 = \{q_0, q_0 + 1\}$, and otherwise set $\mathcal{Q}_0 = \{1\}$. If $S = N - \ell > 0$, define $\mathcal{W}_0 = \{w_0, w_0 + 1\}$, and otherwise set $\mathcal{W}_0 = \{1\}$. Then H is a common pairing if and only if*

$$C_{sq}^T H = H C_{tw} \qquad (q \in \mathcal{Q}_0,\ w \in \mathcal{W}_0).$$

Thus at most four specialization pairs detect the entire residual family. If both residual factors move, the three equations associated with (q_0, w_0), (q_0+1, w_0), and (q_0, w_0+1) already suffice.

Proof. Only the converse requires proof. Put $C_0 = C_{sq_0}$ and $D_0 = C_{tw_0}$. Assume first that $R > 0$. Subtracting the equations for $q_0 + 1$ and q_0 gives

$$\left(C_{s(q_0+1)} - C_{sq_0}\right)^T H = 0.$$

With $\varepsilon \in V_n^\vee$ extracting the coefficient of x^{n-1}, the difference is $-\mathbf{c}_n(s)\varepsilon$, and hence $\mathbf{c}_n(s)^T H = 0$. For $0 \le j < R$, no reduction occurs in $C_0^j \mathbf{c}_n(s) = \mathbf{c}_n(x^j s)$. Using $C_0^T H = H D_0$ therefore gives

$$\mathbf{c}_n(x^j s)^T H = \mathbf{c}_n(s)^T (C_0^T)^j H = \mathbf{c}_n(s)^T H D_0^j = 0.$$

Every difference $C_{sq} - C_0$ has the form $-\mathbf{c}_n(sh)\varepsilon$ with $h \in K[x]_{<R}$, so $C_{sq}^T H = C_0^T H$ for every residual q. The same argument on the right, starting from $H(C_{t(w_0+1)} - C_{tw_0}) = 0$, shows that $HC_{tw} = HD_0$ for every residual w. The endpoint cases simply omit the corresponding difference argument. Combining the two conclusions proves the full family equation. The fourth corner of the two-by-two grid follows from the displayed three equations, which proves the final statement. □

Remark 3.3 (endpoint residual degrees). If one residual degree is zero, the corresponding family contains a single companion matrix. In the proof above the associated truncated ideal is zero, so the descent condition on that side is vacuous; the tensor-product classification remains unchanged. Thus the theorem includes all moving, fixed, and mixed endpoints without a genericity assumption.

Theorem 3.4 (complete rank and radical stratification)**.** *Let $d = \deg g > 0$, and let L_{top} extract the coefficient of x^{d-1} from the reduced representative in $A = K[x]/(g)$. For every $L \in A^\vee$ there is a unique class $a_L \in A$ such that*

$$L(f) = L_{\mathrm{top}}(a_L f) \qquad (f \in A).$$

Choose any polynomial representative a of a_L, put

$$\delta = \gcd(a, g), \qquad e = g/\delta,$$

with monic gcds and the convention $\gcd(0, g) = g$. Then the corresponding common pairing H_L satisfies

$$\operatorname{rank} H_L = \deg e, \qquad \ker_{\mathrm{left}} H_L = M_e^{(n)}, \qquad \ker_{\mathrm{right}} H_L = M_e^{(N)}.$$

Consequently the possible ranks are precisely $\deg e$ as e ranges over the monic divisors of g. In particular, H_L has maximal rank d if and only if a_L is a unit in A.

Proof. The pairing $(f, h) \mapsto L_{\mathrm{top}}(fh)$ is nondegenerate on A: if $0 \neq f$ has reduced degree r, then $L_{\mathrm{top}}(f x^{d-1-r}) \neq 0$. Hence multiplication followed by L_{top} identifies A with $A^\vee$, giving the unique class a_L.

The radical of the induced pairing on A is

$$\{f \in A : L_{\mathrm{top}}(a_L f h) = 0 \ \forall h \in A\} = \operatorname{Ann}_A(a_L),$$

again by nondegeneracy of L_{top}. Write $g = \delta e$ and $a = \delta a_0$ with $\gcd(a_0, e) = 1$. Then $g \mid af$ is equivalent to $e \mid f$, so

$$\operatorname{Ann}_A(a_L) = (e)/(g).$$

The monic gcd $\gcd(a, g)$ is unchanged if a is replaced by another representative of a_L. The ideal $(e)/(g)$ has dimension $d - \deg e$, and therefore the quotient pairing has rank $\deg e$. Pulling the radical back through the two quotient maps from V_n and V_N gives exactly $M_e^{(n)}$ and $M_e^{(N)}$. Every divisor $e \mid g$ occurs by taking $a = g/e$, and the unit criterion is the case $e = g$. □

Corollary 3.5 (the top-coefficient pairing)**.** *The pairing induced directly by L_{top} has rank d and radicals $M_g^{(n)}$ and $M_g^{(N)}$. Hence the maximal common-pairing rank is* $\deg g$.

Corollary 3.6 (finite-field rank strata)**.** *Suppose $K = \mathbb{F}_Q$. For each monic divisor $e \mid g$, the number of common pairings whose left and right radicals are respectively $M_e^{(n)}$ and $M_e^{(N)}$ is*

$$\varphi_Q(e) := \left|(\mathbb{F}_Q[x]/(e))^\times\right| = Q^{\deg e} \prod_{\pi \mid e} \left(1 - Q^{-\deg \pi}\right),$$

where the product is over the distinct monic irreducible divisors of e and $\varphi_Q(1) = 1$. Consequently the number of pairings of rank r is

$$N_r(g) = \sum_{\substack{e \mid g \\ \deg e = r}} \varphi_Q(e),$$

and the rank enumerator is

$$\sum_H z^{\operatorname{rank} H} = \sum_{e \mid g} \varphi_Q(e) z^{\deg e} = \prod_{\pi^{\nu_\pi} \| g} \left(1 + (1 - Q^{-f_\pi}) \sum_{j=1}^{\nu_\pi} Q^{j f_\pi} z^{j f_\pi}\right),$$

where $f_\pi = \deg \pi$. In particular,

$$\sum_{e \mid g} \varphi_Q(e) = Q^{\deg g}.$$

Proof. Under the Frobenius identification in Theorem 3.4, common pairings are parametrized by classes $a \in A = \mathbb{F}_Q[x]/(g)$. The pairing lies in the stratum indexed by e exactly when

$$\gcd(a, g) = g/e.$$

Writing $\delta = g/e$, multiplication by δ gives a bijection

$$(\mathbb{F}_Q[x]/(e))^\times \longrightarrow \{a \in A : \gcd(a, g) = \delta\}, \qquad b \longmapsto \delta b \pmod{g}.$$

The standard unit count in the finite quotient ring gives the product formula. Distinct divisors of the same degree contribute disjoint strata to the same rank, which gives the formula for $N_r(g)$ and the rank enumerator. Summing all strata counts the $Q^{\deg g}$ elements of A. □

Corollary 3.7 (random pairing over a finite field)**.** *Factor*

$$g = \prod_{\pi} \pi^{\nu_\pi}, \qquad f_\pi = \deg \pi,$$

over $\mathbb{F}_Q$*, choose a common pairing uniformly at random, and define its defect from maximal common rank by*

$$D(H) = \deg g - \operatorname{rank} H.$$

Then

$$\Pr(\operatorname{rank} H = \deg g) = \prod_{\pi \mid g} (1 - Q^{-f_\pi}),$$

$$\mathbb{E}D(H) = \sum_{\pi \mid g} f_\pi \sum_{j=1}^{\nu_\pi} Q^{-jf_\pi},$$

and

$$\operatorname{Var} D(H) = \sum_{\pi \mid g} f_\pi^2 \left[\sum_{j=1}^{\nu_\pi} (2j-1) Q^{-jf_\pi} - \left(\sum_{j=1}^{\nu_\pi} Q^{-jf_\pi} \right)^2 \right].$$

Proof. Maximal rank is equivalent to the parameter $a \in \mathbb{F}_Q[x]/(g)$ being a unit, giving the first formula. Under the Chinese remainder theorem, the truncated π-adic valuations of a uniformly random class are independent; we set $v_\pi(0) = \nu_\pi$ in the factor $\mathbb{F}_Q[x]/(\pi^{\nu_\pi})$. Then

$$\Pr(v_\pi(a) \geq j) = Q^{-jf_\pi} \qquad (1 \leq j \leq \nu_\pi).$$

The maximal-rank defect contributed by this factor is $f_\pi v_\pi(a)$. Taking expectations and summing the tail probabilities proves the second formula. For a nonnegative integer-valued random variable V bounded by ν,

$$V^2 = \sum_{j=1}^{\nu} (2j-1) \mathbf{1}_{\{V \geq j\}}.$$

Applying this identity to each independent truncated valuation gives its second moment. Subtracting the square of the mean and summing the independent contributions proves the variance formula. □

Corollary 3.8 (common bilinear symmetrizing forms)**.** *For one family $\mathcal{F}_n(s)$, every matrix satisfying*

$$C_{sq}^T H = H C_{sq} \qquad \forall q$$

is of the form

$$H_{ij} = L(x^{i+j} \bmod s), \qquad L \in (K[x]/(s))^\vee.$$

No symmetry hypothesis is needed: every common solution is automatically a symmetric Hankel matrix. The solution space has dimension m, its maximal rank is m, and no common form is nondegenerate when $m < n$.

Corollary 3.9 (split squarefree basis)**.** *Suppose g splits over K with distinct roots $\rho_1, \dots, \rho_d$. Then every common pairing is*

$$H = \sum_{\alpha=1}^{d} \lambda_\alpha v_n(\rho_\alpha) v_N(\rho_\alpha)^T, \qquad v_r(\rho) = (1, \rho, \dots, \rho^{r-1})^T.$$

Moreover $\operatorname{rank} H$ *is the number of nonzero coefficients λ_α.*

Remark 3.10 (repeated roots). If K has characteristic zero and $g = \prod_\alpha (x - \rho_\alpha)^{e_\alpha}$ splits, the dual of $K[x]/(g)$ is spanned by the jet functionals $f \mapsto f^{(j)}(\rho_\alpha)$ for $0 \le j < e_\alpha$. This gives a confluent Vandermonde basis for the common pairing space. In positive characteristic, the same statement is expressed with Hasse derivatives.

4. Common invariant subspaces

We now return to one family $\mathcal{F}_n(s)$.

Theorem 4.1 (fixed-factor invariant lattice)**.** *The common invariant subspaces of $\mathcal{F}_n(s)$ are*

$$0 \quad \text{and} \quad M_d^{(n)}$$

for monic divisors $d \mid s$. If $\deg s = n$, then $0 = M_s^{(n)}$ is already included. For divisors $a, b \mid s$,

$$M_a^{(n)} + M_b^{(n)} = M_{\gcd(a,b)}^{(n)}, \qquad M_a^{(n)} \cap M_b^{(n)} = M_{\operatorname{lcm}(a,b)}^{(n)}.$$

Thus the common invariant-subspace lattice is the divisor lattice of s with its order reversed.

Proof. For one fixed p, the C_p-invariant subspaces are the $K[x]$-submodules of $K[x]/(p)$, hence the ideals $(d)/(p)$ with $d \mid p$. In degree-$< n$ representatives, these are precisely the spaces $M_d^{(n)}$.

Let d be monic with $\deg d < n$, so that $M_d^{(n)} e 0$. For another monic polynomial p' of degree n, the space $M_d^{(n)}$ is $C_{p'}$-invariant if and only if $d \mid p'$. Indeed, every shift is visibly inside $M_d^{(n)}$ except the last one. Reducing $x^{n-\deg d} d$ modulo p' remains divisible by d exactly when p' is divisible by d. The zero space is, of course, invariant for every p'.

Therefore a nonzero subspace common to all $p_q = sq$ is $M_d^{(n)}$ for a monic d dividing every sq. If $m = n$, the residual is 1 and this already means $d \mid s$. If $m < n$, put $R = n - m$ and use the two residual factors x^R and $x^R + 1$, which are coprime over every field. A polynomial dividing both sx^R and $s(x^R + 1)$ must divide s. The converse is immediate. The sum and intersection formulas are the corresponding ideal identities in the PID $K[x]$. □

5. The algebra generated by the residual family

Let

$$\mathcal{A}_s = K\langle C_{sq} : q \text{ monic of degree } n - m\rangle \subseteq \mathrm{End}_K(V_n)$$

be the unital algebra generated by the entire family, and set

$$R = n - m, \qquad M_s = M_s^{(n)}.$$

Theorem 5.1 (generated algebra)**.** *Choose one residual factor q_0 and put $C_0 = C_{sq_0}$. Then*

$$\mathcal{A}_s = \mathrm{Hom}_K(V_n, M_s) + \mathrm{span}_K\{I, C_0, \dots, C_0^{m-1}\},$$

where the second summand is omitted when $m = 0$. Equivalently,

$$\mathcal{A}_s = \{T \in \mathrm{End}_K(V_n) : T(M_s) \subseteq M_s,\ \overline{T} \in K[\overline{C}_0] \cong K[x]/(s)\},$$

where bars denote the induced maps on V_n/M_s. The sum is direct and

$$\dim_K \mathcal{A}_s = n(n - m) + m.$$

In particular, $s = 1$ gives $\mathcal{A}_s = \mathrm{End}_K(V_n)$, while $m = n$ gives the cyclic algebra $\mathcal{A}_s = K[C_s] \cong K[x]/(s)$.

Proof. Let $\varepsilon \in V_n^\vee$ extract the coefficient of x^{n-1}. If $h \in K[x]_{<R}$, changing q_0 to $q_0 + h$ changes only the final companion column and gives

$$C_{s(q_0+h)} - C_{sq_0} = -\mathbf{c}_n(sh)\,\varepsilon.$$

As h varies, the vectors $\mathbf{c}_n(sh)$ fill M_s. Hence $\mathcal{A}_s$ contains every rank-one map $u\varepsilon$ with $u \in M_s$.

The rows

$$\varepsilon, \varepsilon C_0, \ldots, \varepsilon C_0^{n-1}$$

form a basis of $V_n^\vee$. In the monomial basis they are triangular, in reverse coordinate order, with diagonal entries 1. Right multiplication by powers of C_0 therefore produces every rank-one map $u\varphi$ with $u \in M_s$ and $\varphi \in V_n^\vee$. Thus $\mathrm{Hom}_K(V_n, M_s) \subseteq \mathcal{A}_s$.

Every family member preserves M_s, and all induce multiplication by x on $V_n/M_s \cong K[x]/(s)$. Hence the image of $\mathcal{A}_s$ on the quotient is contained in $K[\overline{C}_0]$. Conversely, let

$$\rho : \{T \in \mathrm{End}_K(V_n) : T(M_s) \subseteq M_s\} \longrightarrow \mathrm{End}_K(V_n/M_s)$$

be the quotient representation. Its kernel consists exactly of the maps whose image is contained in M_s, namely $\mathrm{Hom}_K(V_n, M_s)$. The powers $I, C_0, \ldots, C_0^{m-1}$ lift a basis of $K[\overline{C}_0] \cong K[x]/(s)$. This proves the structural description.

If a polynomial $a(C_0)$ with $\deg a < m$ has image contained in M_s, then its induced multiplication map on $K[x]/(s)$ is zero. Thus $s \mid a$, forcing $a = 0$. The sum is direct, and

$$\dim \mathcal{A}_s = n \dim M_s + m = nR + m.$$

□

Corollary 5.2 (two companion matrices suffice)**.** *Assume $m < n$ and fix any monic residual factor q_0 of degree $R = n - m$. Then*

$$\mathcal{A}_s = K\langle C_{sq_0}, C_{s(q_0+1)}\rangle.$$

If $n \geq 2$, two family members are minimal: no single matrix can generate $\mathcal{A}_s$ as a unital algebra.

Proof. The difference of the two generators is the rank-one map

$$C_{s(q_0+1)} - C_{sq_0} = -\mathbf{c}_n(s)\,\varepsilon.$$

For $0 \le j < R$, no reduction occurs in $C_{sq_0}^j s = x^j s$, so left multiplication by the powers of C_{sq_0} produces $x^j s\,\varepsilon$. These vectors form a basis of M_s. Right multiplication by the powers of C_{sq_0} then produces every map in $\mathrm{Hom}_K(V_n, M_s)$ exactly as in the proof of Theorem 5.1; the quotient powers provide the remaining summand. Hence the two matrices generate $\mathcal{A}_s$.

A unital algebra generated by one $n \times n$ matrix has dimension at most n by Cayley–Hamilton. For $m < n$ and $n \ge 2$,

$$\dim \mathcal{A}_s - n = n(n-m) + m - n = (n-m)(n-1) > 0,$$

so one generator cannot suffice. □

Corollary 5.3 (triangular normal form)**.** *Choose a vector-space splitting $V_n \cong K[x]/(s) \oplus M_s$. Then*

$$\mathcal{A}_s \cong \begin{pmatrix} K[x]/(s) & 0 \\ \mathrm{Hom}_K(K[x]/(s), M_s) & \mathrm{End}_K(M_s) \end{pmatrix}.$$

For $R > 0$, the center of $\mathcal{A}_s$ is K. For $R = 0$, the algebra is commutative and equals $K[x]/(s)$.

Proof. The block form is a restatement of Theorem 5.1. If $R > 0$ and $m = 0$, then $\mathcal{A}_s = \mathrm{End}_K(V_n)$ and the center is K. If $R > 0$ and $m > 0$, a central element must commute with the lower-right full matrix algebra and with all lower-left maps. Its lower-right block is therefore scalar, its off-diagonal block vanishes, and the identity $\lambda L = Lb$ for every $L \in \mathrm{Hom}_K(K[x]/(s), M_s)$ forces $b = \lambda I$. The fixed endpoint is immediate. □

Corollary 5.4 (Jacobson radical and semisimple quotient)**.** *Let*

$$s_{\mathrm{red}} = \prod_{\pi \mid s} \pi$$

be the squarefree part of s, with the product over the distinct monic irreducible factors and with $s_{\mathrm{red}} = 1$ when $s = 1$. If $R > 0$, then relative to the splitting in Theorem 5.3,

$$J(\mathcal{A}_s) \cong \begin{pmatrix} (s_{\mathrm{red}})/(s) & 0 \\ \mathrm{Hom}_K(K[x]/(s), M_s) & 0 \end{pmatrix},$$

and

$$\mathcal{A}_s/J(\mathcal{A}_s) \cong K[x]/(s_{\mathrm{red}}) \times M_R(K).$$

If $R = 0$, then $J(\mathcal{A}_s) = (s_{\mathrm{red}})/(s)$ and $\mathcal{A}_s/J(\mathcal{A}_s) \cong K[x]/(s_{\mathrm{red}})$. Consequently, $\mathcal{A}_s$ is semisimple exactly when either $s = 1$ or $R = 0$ and s is squarefree.

Proof. Put $B = K[x]/(s)$, $M = \mathrm{Hom}_K(K[x]/(s), M_s)$, and $D = \mathrm{End}_K(M_s) \cong M_R(K)$. The homomorphism

$$\begin{pmatrix} B & 0 \\ M & D \end{pmatrix} \longrightarrow B/J(B) \times D, \qquad \begin{pmatrix} b & 0 \\ u & d \end{pmatrix} \longmapsto (b + J(B), d),$$

has kernel $\left(\begin{smallmatrix} J(B) & 0 \\ M & 0 \end{smallmatrix}\right)$ and semisimple image. Since B is finite-dimensional and commutative, $J(B)$ is nilpotent. If $J(B)^k = 0$, then a product of $k+1$ elements of the displayed kernel is zero: after the first multiplication the lower-left entry contains a factor from $J(B)$, and after k such factors it vanishes. Hence the kernel is a nilpotent ideal and is contained in the Jacobson radical. Because the quotient by this kernel is semisimple, the reverse containment also holds. Thus the kernel is the full radical. Finally, $J(B) = (s_{\mathrm{red}})/(s)$, which proves both displayed formulas. When $R = 0$ only B remains. If $R > 0$ and $m > 0$, the lower-left bimodule is nonzero, so the radical is nonzero; the remaining cases give the semisimplicity criterion. □

Corollary 5.5 (common commutant)**.** *If $m < n$, then*

$$\{X \in \mathrm{End}_K(V_n) : XC_{sq} = C_{sq}X \ \forall q\} = KI.$$

If $m = n$, the family has one member and its commutant is $K[C_s] \cong K[x]/(s)$.

Proof. The common commutant is the commutant of $\mathcal{A}_s$. In the moving case, commute first with all maps in $\mathrm{Hom}_K(V_n, M_s)$. Relative to the splitting in Theorem 5.3, the full lower-right matrix algebra and all lower-left maps force every commuting endomorphism to be a scalar on both summands with the same scalar. The case $m = 0$ is included because then $\mathrm{Hom}_K(V_n, M_s) = \mathrm{End}_K(V_n)$. The fixed case is the standard commutant of a cyclic operator. □

6. Common covariant intertwiners

The transpose in the pairing equation is essential. Without it, motion in the left-hand, or target, family has a stronger effect.

Theorem 6.1 (asymmetry of common intertwiners)**.** *Let $X \in K^{n\times N}$ satisfy*

$$C_{sq}X = XC_{tw} \qquad \textit{for every residual pair } (q, w).$$

Thus X maps the source module of C_{tw} to the target module of C_{sq}. If $m < n$, so the target family moves, then $X = 0$ over every field K.

If $m = n$, so the target module $K[x]/(s)$ is fixed, then the common intertwiners are multiplication maps by elements of

$$\mathrm{Ann}_{K[x]/(s)}(t).$$

Writing $g = \gcd(s,t)$ and $s = gs_0$, this annihilator is

$$(s_0)/(s) \cong K[x]/(g).$$

Hence the fixed-target common intertwiner space has dimension $\deg g$.

Proof. Suppose first that $R = n - m > 0$. Fix a monic residual factor q_0 of degree R and any source residual factor w_0. Since both q_0 and $q_0 + 1$ are monic of degree R, subtracting their intertwining equations gives

$$\big(C_{s(q_0+1)} - C_{sq_0}\big)X = 0.$$

Let $\varepsilon \in V_n^\vee$ extract the coefficient of x^{n-1}. The difference of the two companion matrices is the nonzero rank-one map

$$C_{s(q_0+1)} - C_{sq_0} = -\mathbf{c}_n(s)\,\varepsilon,$$

so $\varepsilon X = 0$. Put $C_0 = C_{sq_0}$ and $D_0 = C_{tw_0}$. From $C_0X = XD_0$ one obtains

$$\varepsilon C_0^j X = \varepsilon X D_0^j = 0 \qquad (0 \le j < n).$$

The rows $\varepsilon, \varepsilon C_0, \dots, \varepsilon C_0^{n-1}$ form a basis of $V_n^\vee$: in the reversed monomial basis their coefficient matrix is triangular with diagonal entries one. Hence $X = 0$. No assumption on the cardinality or characteristic of K is used.

Now suppose $m = n$. The first $N - 1$ companion columns give $X(x^j) = x^j X(1)$ for $0 \le j < N$, with multiplication taken in $K[x]/(s)$. Thus an intertwiner into the fixed target module $K[x]/(s)$ is determined by $a = X(1)$, and the condition for the source specialization tw is

$$twa = 0 \quad \text{in } K[x]/(s).$$

If the source residual degree is positive, subtracting the conditions for $w = x^r$ and $w = x^r + 1$ gives $ta = 0$; for degree zero this is already the condition. Conversely, $ta = 0$ implies $twa = 0$ for every residual factor w, so the common space is exactly $\mathrm{Ann}_{K[x]/(s)}(t)$. Write $s = gs_0$ and $t = gt_0$ with $\gcd(s_0, t_0) = 1$. The divisibility $s \mid ta$ is equivalent to $s_0 \mid a$, so the annihilator is $(s_0)/(s)$ and is identified with $K[x]/(g)$ by $h \mapsto s_0 h$. □

Corollary 6.2 (complete fixed-target intertwiner stratification)**.** *Assume $m = n$, write $g = \gcd(s, t)$ and $s = gs_0$, and identify the common intertwiner space with $K[x]/(g)$ by*

$$b \longmapsto X_b, \qquad X_b(h \bmod tw) = s_0 bh \bmod s.$$

Choose a polynomial representative b, put

$$\delta = \gcd(b, g), \qquad e = g/\delta,$$

with $\gcd(0, g) = g$. Then

$$\operatorname{rank} X_b = \deg e, \qquad \ker X_b = M_e^{(N)}, \qquad \operatorname{im} X_b = M_{s_0\delta}^{(n)}.$$

Consequently, the possible intertwiner ranks are exactly the degrees of monic divisors of g. The maximal rank is $\deg g$ and occurs precisely when b is a unit modulo g; in particular, multiplication by s_0 has kernel $M_g^{(N)}$ and image $M_{s_0}^{(n)}$.

Proof. The kernel condition is

$$s \mid s_0 bh \quad \Longleftrightarrow \quad g \mid bh \quad \Longleftrightarrow \quad e \mid h,$$

where the last equivalence uses $b = \delta b_0$ and $\gcd(b_0, e) = 1$. Hence the kernel is $M_e^{(N)}$ and the rank is $\deg e$. The image is the ideal generated by $s_0 b$ in $K[x]/(s)$. Since $\gcd(s_0 b, s) = s_0\delta$, its degree-$< n$ representatives are exactly $M_{s_0\delta}^{(n)}$. The remaining statements follow as in Theorem 3.4. □

Corollary 6.3 (finite-field intertwiner ranks)**.** *In the fixed-target case over $K = \mathbb{F}_Q$, the number of common intertwiners in the stratum indexed by $e \mid g$ is $\varphi_Q(e)$. Hence the rank enumerator and maximal-rank probability are the formulas in Theorems 3.6 and 3.7; the same is true of the mean and variance of the maximal-rank defect $\deg g - \operatorname{rank} X$.*

Proof. Under the parameterization $b \in \mathbb{F}_Q[x]/(g)$, the stratum indexed by e is characterized by $\gcd(b, g) = g/e$. The same unit-group bijection used in Theorem 3.6 applies verbatim. □

7. Examples

Example 7.1 (one fixed root). Let $s = x - \rho$. Every common symmetrizing form is a scalar multiple of

$$v_n(\rho)v_n(\rho)^T.$$

The only nonzero proper common invariant subspace is $M^{(n)}_{x-\rho}$, and the generated algebra has dimension $n(n-1)+1$. Thus a single fixed root leaves one rank-one common form and one forced invariant hyperplane, while the residual family generates every map into that hyperplane.

Example 7.2 (a two-matrix intermediate algebra). Let $n = 4$ and $s = (x-1)(x+1)$, so $m = 2$ and $R = 2$. For any monic quadratic q_0, the two matrices C_{sq_0} and $C_{s(q_0+1)}$ generate a 10-dimensional algebra. It is neither the 4-dimensional cyclic algebra of one companion matrix nor the full 16-dimensional matrix algebra; its block form is

$$\begin{pmatrix} K[x]/(s) & 0 \\ M_{2\times 2}(K) & M_2(K) \end{pmatrix}.$$

This is the exact intermediate algebra forced by the common factor s.

Example 7.3 (coprime fixed factors). If $\gcd(s,t) = 1$, there is no nonzero common pairing between the two families. This remains true even though each individual pair of companion matrices may admit nonzero solutions of $C_{sq}^T H = HC_{tw}$.

Example 7.4 (repeated fixed factor). Let $s = (x-\rho)^e$. The common symmetrizing forms have dimension e and are spanned in characteristic zero by the first e jet functionals at ρ. The common invariant subspaces form the chain

$$V_n \supset M^{(n)}_{x-\rho} \supset \cdots \supset M^{(n)}_{(x-\rho)^e}.$$

When a fixed target and source have gcd $(x-\rho)^e$, their common intertwiners have every rank $0, 1, \ldots, e$.

8. Relation to previous work and scope

The fixed-polynomial facts used in the proofs are classical. Fuhrmann's polynomial-model viewpoint treats a cyclic operator through the quotient $K[x]/(p)$ and develops the associated duality and intertwining machinery [1, 2]; in that sense the module language of the present paper has established

ancestry. A companion matrix is a cyclic realization of $K[x]/(p)$, its invariant subspaces are ideals in that quotient, and Hankel or Bezoutian forms are naturally linked to companion relations [3, 4, 5]. Simultaneous reduction problems for sets of companion matrices were studied by Bart and Thijsse [6], and Zuidwijk related simultaneous similarity of a pair of companion matrices to their transposes to the associated Bezoutian [7]. General matrix symmetrizers are also classical objects [8]. These results concern fixed finite collections or invertible similarities. By contrast, Theorem 3.8 classifies all forms, including singular ones, that are shared by every member of a residual family. The rank defect of a multiplication matrix $f(C_g)$ is controlled by $\gcd(f, g)$, with stronger Smith-form statements in companion rings proved by Noferini and Williams [9]. Accordingly, the single-pair model, the quotient multiplication rank inside Theorem 3.4, and the general cyclic-module dictionary are not claimed as new. The theorem packages those ingredients with a family-uniform quantifier and identifies the exact left and right pullback radicals for unequal ambient dimensions.

The common-factor criterion for invariant subspaces of several companion matrices is known [10]. Moreover, Guersenzvaig and Szechtman proved that two companion matrices with polynomial gcd of degree m generate, over an integrally closed base, a free module of rank $n + (n - m)(n - 1)$ and supplied a presentation [11]. Thus the dimension in Theorem 5.1 and the fact that two suitable members suffice are prior phenomena, not novelty claims here.

The family-uniform results established here are therefore statements about the structures surviving intersection over every allowed residual deformation:

- two independently moving residual families retain exactly the dual of $K[x]/(\gcd(s, t))$ in the common rectangular bilinear equation, including the fixed residual endpoint cases;
- the quotient-ring multiplication data pull back to the exact ambient left and right radicals in Theorem 3.4, uniformly for unequal family sizes, and yield the complete finite-field rank distribution;
- the full common-pairing condition is detected by two residual points on each moving side, as in Theorem 3.2;
- the known two-companion dimension admits the residual-family inverse-image and triangular description in Theorems 5.1 and 5.3, with the radical and semisimple quotient made explicit in Theorem 5.4;

- fixed-target and moving-target common intertwiners exhibit the field-independent sharp asymmetry described in Theorem 6.1, and the fixed-target side has the complete kernel, image, and finite-field rank stratification of Theorems 6.2 and 6.3.

The paper does not address arbitrary companion forms, block companion linearizations, or structured matrix polynomials. It does not assert that the generated-algebra section is independent of the established two-matrix literature; that section is retained because it connects the simultaneous pairing and invariant structures to the exact algebra acting on the family. The classification may be summarized in one sentence: the common bilinear structure descends to a gcd quotient, the common invariant lattice descends to divisors of the fixed factor, the generated algebra is full below that quotient, and a moving target destroys all common covariant intertwiners.

9. Reproducibility

Online Resource 1 contains two exact-arithmetic verification programs. The script `supplement/verify.py` constructs Frobenius companion matrices over the rationals and checks, for squarefree and repeated-factor examples, the common-pairing dimension $\deg \gcd(s,t)$, the two-point detection principle, the divisor-controlled rank and radical stratification, the predicted common symmetrizers and fixed-factor invariant ideals, the generated-algebra dimension $n(n-m)+m$, the two-generator corollary, scalar common commutants in moving cases, and the complete fixed-target intertwiner rank, kernel, and image stratification. A second script, `supplement/audit_companion_linear_systems.py`, enumerates full residual families over $\mathbb{F}_2$, solves the raw simultaneous pairing and intertwiner equations, checks the three-equation detection certificate, recovers pairing and intertwiner kernel strata by brute force, closes the generated algebra, reconstructs its Jacobson radical from the finite-ring unit criterion, and enumerates every subspace in a small invariant-lattice example. It does not call the quotient classification routines in the primary verifier. The verification programs were developed with AI-assisted coding support and then reviewed, tested, and validated by the author. These computations are verification aids; the proofs above are independent of them.

CRediT authorship contribution statement

Aaron Nagy: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review and editing.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data and code availability

No external data were used. Online Resource 1 constructs the companion families directly and checks the common-pairing dimensions, two-point reduction, rank and radical strata, finite-field counts, invariant subspaces, generated algebras, commutants, and full fixed-target intertwiner strata in exact small cases. A separate direct finite-field audit solves the simultaneous equations, enumerates the rank and kernel strata, and reconstructs the Jacobson radical without using the quotient classifications stated in the paper. The proofs do not depend on a finite computational cutoff. The code is supplied with the submission as Online Resource 1.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI ChatGPT and Codex to assist with conjecture exploration, symbolic checks, verification-code development, literature-query formulation, manuscript organization, and language editing. After using these tools, the author independently re-derived and reviewed the mathematical arguments, verified citations against publisher or DOI metadata, reviewed and tested the code, and edited all AI-assisted text. No AI-generated output was accepted as a proof or bibliographic authority. The author takes full responsibility for the content of the publication.

References

[1] Paul A. Fuhrmann, Duality in polynomial models with some applications to geometric control theory, *IEEE Transactions on Automatic Control* 26 (1981), 284–295. doi:10.1109/TAC.1981.1102570.

[2] Paul A. Fuhrmann, *A Polynomial Approach to Linear Algebra*, second edition, Universitext, Springer, New York, 2012. doi:10.1007/978-1-4614-0338-8.

[3] Miroslav Fiedler, Polynomials and Hankel matrices, *Linear Algebra and its Applications* 66 (1985), 235–248. doi:10.1016/0024-3795(85)90135-1.

[4] Minghua Lin and Harald K. Wimmer, Bilinear characterizations of companion matrices, *Special Matrices* 2 (2014), 99–105. doi:10.2478/spma-2014-0010.

[5] Yousong Luo and Robin Hill, Companion matrices and their relations to Toeplitz and Hankel matrices, *Special Matrices* 3 (2015), 214–226. doi:10.1515/spma-2015-0021.

[6] H. Bart and G. Ph. A. Thijsse, Simultaneous reduction to companion and triangular forms of sets of matrices, *Linear and Multilinear Algebra* 26 (1990), 231–241. doi:10.1080/03081089008817981.

[7] R. A. Zuidwijk, Simultaneous similarity of pairs of companions to their transposes, in A. Gheondea, R. N. Gologan, and D. Timotin (eds.), *Operator Theory and Analysis*, Operator Theory: Advances and Applications 122, Birkhäuser, Basel, 2001, 417–425. doi:10.1007/978-3-0348-8283-5_16.

[8] Froilán M. Dopico and Frank Uhlig, Computing matrix symmetrizers, part 2: New methods using eigendata and linear means; a comparison, *Linear Algebra and its Applications* 504 (2016), 590–622. doi:10.1016/j.laa.2015.06.031.

[9] Vanni Noferini and Gerald Williams, Smith forms of matrices in companion rings, with group theoretic and topological applications, *Linear Algebra and its Applications* 708 (2025), 372–404. doi:10.1016/j.laa.2024.12.003.

[10] Natalio H. Guersenzvaig and Fernando Szechtman, Subalgebras of matrix algebras generated by companion matrices, *Linear Algebra and its Applications* 432 (2010), 2691–2700. doi:10.1016/j.laa.2009.12.019.

[11] Natalio H. Guersenzvaig and Fernando Szechtman, Is every matrix similar to a polynomial in a companion matrix?, *Linear Algebra and its Applications* 437 (2012), 1611–1627. doi:10.1016/j.laa.2012.04.049.